ON STABILIZATION OF NONSTATIONARY SYSTEMS

(O STABILIZATSII NESTATSIONARNYKH SISTEM)

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The general problem of asymptotic stabilization of steady-state motions of nonlinear control systems [1] was examined in [2]. In this paper, conditions of stability are established in the first approximation for nonstationary systems in one particular case.

We examine the following control system:

$$dy / dt = f(t, y, \omega) \qquad (y \in \{R^n\}, \omega \in \{R^m\}) \tag{1}$$

where f is a given vector function, y is the vector of phase coordinates of the system. Vector ω is the control which we consider unaffected by disturbances. Vector y is subject to small perturbations x, so that in (1)

$$y(t) = y^*(t) + x(t) \tag{2}$$

Here $y^*(t)$ is a given motion generated by the control $\omega^*(t)$. We let

$$u = \omega - \omega^*(t) \tag{3}$$

Substituting (2) and (3) into Equation (1) and expanding the right-hand side with respect to quantities x and u we obtain equations of perturbed motion

$$\frac{dx}{dt} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i} + \sum_{j=1}^{m} \frac{\partial f}{\partial \omega_j}\frac{\partial \omega_j^*}{\partial y_i}\right) x_i + \sum_{j=1}^{m} \frac{\partial f}{\partial \omega_j} u_j + g(t, x, u) \tag{4}$$

where derivatives are computed along the motion $y = y^*(t)$; $g(t, x, u)$ designates terms the order of which with respect to x and u is uniformly higher than first in t for $0 \leqslant t \leqslant \infty$, i.e. we assume that the following inequality is fulfilled

$$\|g(t, x, u)\| \leqslant N [\|x\| + \|u\|]^{1+\alpha} \qquad (N = \text{const} > 0, \quad \alpha = \text{const} > 0) \tag{5}$$

Symbol $\|q\|$ designates Euclidean norm of vector $q = \{q_1, \ldots, q_k\}$

$$\|q\| = \sqrt{q_1^2 + \ldots + q_k^2}$$

If for $u = 0$ the zeroth solution of system (4) is unstable, the problem of stabilization of motion (1) arises, i.e. the problem of selecting such a function $u(t, x)$ that on substitution of this function in (4) the zeroth solution $x = 0$ would be asymptotically stable according to Liapunov [1]. Thus we shall examine the following system:

$$dx / dt = A(t) x + B(t) u + g(t, x, u) \tag{6}$$

where $A(t)$ is an $n \times n$ matrix, $B(t)$ is an $n \times m$ matrix, u is m-vector and g is a vector-function which satisfies inequality (5). In detailed notation system (6) has the form

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)\,x_j + \sum_{k=1}^{m} b_{ik}(t)\,u_k + g_i(t, x, u) \qquad (i = 1, \ldots, n) \tag{7}$$

Together with the complete system (6) we shall examine the system of first approximation

$$dx \,/\, dt = A\,(t)\,x + B\,(t)\,u \tag{8}$$

We assume here that elements $a_{ij}(t)$ and $b_{ik}(t)$ of matrices $A(t)$ and $B(t)$ have time derivatives da_{ij}/dt and db_{ik}/dt . We limit ourselves to the examination of the case where for each fixed value $t = \tau =$ const > 0 the rank of the matix

$$V = \{B\,(\tau), A\,(\tau)\,B\,(\tau), \ldots, A^{n-1}\,(\tau)\,B\,(\tau)\} \tag{9}$$

is equal to (n)

$$r\,(V) = n \tag{10}$$

Sufficient conditions will be established below for which the unperturbed motion of system (6) is stabilized by linear control

$$u = P\,(t)\,x, \quad \text{for} \quad u_k(t, x) = \sum_{j=1}^{n} p_{kj}(t)\,x_j \quad (k = 1, \ldots, m) \tag{11}$$

independently of members $g(t, x, u)$.

Let us examine matrix (9). We select any n columns $l^{(i)}$ from this matrix and construct the quadratic form from some variable λ_i

$$\theta\,(\lambda) = \sum_{i,j=1}^{n} (l^{(i)}\,(\tau)\cdot l^{(j)}\,(\tau))\,\lambda_i\lambda_j \tag{12}$$

Here the symbol $(l^{(i)}\,(\tau)\cdot l^{(j)}(\tau))$ designates the scalar product of vectors $l^{(i)}$ and $l^{(j)}$. The form (12) will play a fundamental role in the criterion of stabilization established below.

T h e o r e m . If for any $\tau \geqslant 0$ in matrix (8) it is possible to select n linearly independent columns $l^{(1)}, \ldots, l^{(n)}$ so that the quadratic form (12) should be positive definite, then we can find a constant $\gamma > 0$ such that when inequalities

$$\left|\frac{da_{ij}\,(t)}{dt}\right| \leqslant \gamma, \qquad \left|\frac{db_{ik}\,(t)}{dt}\right| \leqslant \gamma \tag{13}$$

are satisfied, the unperturbed motion of system (6) can be stabilized by the linear control (11) independently of terms $g(t, x, u)$.

P r o o f . Let us examine the system with constant coefficients

$$dx \,/\, dt = A\,(\tau)\,x + B\,(\tau)\,u \tag{14}$$

where $\tau \geqslant 0$ is a fixed number. This system satisfies the condition of stabilization given in Theorem 4.1 [2] (see also papers [3 to 5]). In fact, the space $\{W^\tau\}$ which is mentioned in Theorem 4.1, coincides according to (10) with the space $\{x_i\}$ and thus all eigenvectors $S^{+}_{(i)}$ and $S_{(k)}$ of matrix $A(\tau)$ in case of its simple structure or vectors $T^{+}_{(i)}$ and $T_{(k)}$ in the general case (see [2], pp.997 to 999) automatically fall into space $\{W^\tau\}$. Consequently, by virtue of Theorem 4.1 a linear control exists

$$u\,(\tau, x) = P\,(\tau)\,x \tag{15}$$

such that for every $\tau \geqslant 0$ the trivial solution of the system of linear equations with constant coefficients

$$dx \,/\, dt = A\,(\tau)\,x + B\,(\tau)\,P\,(\tau)\,x \tag{16}$$

will be automatically stable.

According to [6] (p.62) a positive definite Liapunov's function exists for asymptotically stable system (16)

$$v(\tau, x) = \sum_{i,j=1}^{n} \alpha_{ij}(\tau) x_i x_j \qquad (17)$$

such that

$$\left(\frac{dv(\tau, x(t))}{dt}\right)_{(16)} = -\sum_{i=1}^{n} x_i^2(t) \qquad (\tau = \text{const}) \qquad (18)$$

Coefficients of this function $\alpha_{ij}(\tau)$, as is well known, are computed from conditions (18). For determination of these coefficients([6] pp.57-66) a linear system of algebraic equations which depend on $a_{ij}(\tau)$, $b_{ik}(\tau)$ and $p_{kj}(\tau)$ is obtained.

Here it is important to note the following. Control (15) under the condition of positive definiteness of form (12), can be selected so that matrix $P(\tau)$ will be uniformly bounded for $\tau \geqslant 0$, while form (17) in this case will have bounded coefficients for all $\tau \geqslant 0$ and will be positive definite uniformly with respect to τ. The validity of these statements is derived on the basis of known estimates of control theory of linear systems (8). In this connection values $p_{kj}(t)$ can be computed by solving the problem of analytical design of the control for system (16) [7] (see note 3.3 [2],p.994). Then we can select a control $u(\tau, x) = P(\tau)x$ so that for motions of systems (16), the following inequalities

$$\| x(t) \| \leqslant \beta \| x(t_0) \| e^{-\alpha(t-t_0)} (\alpha, \beta = \text{const}, \alpha > 0, \beta > 0)$$

would be satisfied uniformly with respect to τ.

Now we compute the derivative dv/dt by virtue of system (16) assuming τ in quadratic form (17) and in system (16) to be a variable quantity equal to t. We have

$$\left(\frac{dv(t, x(t))}{dt}\right)_{(16)} = \left(\frac{dv(\tau, x(t))}{dt}\right)_{(16)} + \frac{\partial v(t, x(t))}{\partial t} \qquad \left(\frac{\partial v}{\partial t} = \sum_{i,j=1}^{n} \frac{d\alpha_{ij}(t)}{dt} x_i x_j\right) \qquad (19)$$

As was noted above, quantities $\alpha_{ij}(\tau)$ are computed from linear equations, coefficients of which depend on $a_{ij}(\tau)$, $b_{ik}(\tau)$ and $p_{kj}(\tau)$. For the condition of positive definiteness of form (12) the determinant Δ of this system is uniformly different from zero [8], i.e.

$$|\Delta| > \nu \qquad (\nu = \text{const}, \nu > 0)$$

It follows from this that if derivatives $da_{ij}(t)/dt$ $db_{ik}(t)/dt$ and $dp_{kj}(t)/dt$ are small, then derivatives $d\alpha_{ij}(t)/dt$ will also be small. However, quantities $da_{ij}(t)/dt$ and $db_{jk}(t)/dt$ are selected small according to condition (13). Smallness of quantities $dp_{kj}(t)/dt$ also follows from smallness of quantities $da_{ij}(t)/dt$ and $db_{kj}(t)/dt$. In fact, as was noted above, quantities $p_{kj}(t)$ can be computed by solving the problem of analytical design of the control [7] for system (14). It follows from the theory of this problem that under the condition of positive-definiteness of quadratic form (12), the quantities $dp_{kj}(t)/dt$ exist and are small if only the quantities $da_{ij}(t)/dt$ and $db_{ik}(t)/dt$ are small.

Thus the second term in (19) can be made small in comparison to the first by selection of the quantity $\gamma > 0$. It follows from this that the derivative $(dv(t, x(t))/dt)_{(16)}$ for sufficiently small γ, is a negative definite quadratic form from x_i. Consequently the quadratic form $v(t, x)$, defined in (17), satisfies the following conditions:

$$c_1 \| x \|^2 \leqslant v(t, x) \leqslant c_2 \| x \|^2, \qquad \left| \frac{\partial v}{\partial x_i} \right| \leqslant c_3 \| x \|$$

Here c_1, c_2 and c_3 are constants independent of t. The derivative of this function $(dv/dt)_{(8)}$ for $u = P(t)x$ is a negative-definite function.

We construct the derivative from the form $v(t, x)$ by virtue of the complete system (6) for $u = P(t)x$.

We have

$$\left(\frac{dv}{dt}\right)_{(6)} = \left(\frac{dv}{dt}\right)_{(8)} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} g_i(t, x, u) \qquad (20)$$

By virtue of uniform boundedness of partial derivatives $\partial v/\partial x_i$, the quantity (20) is also a negative definite function for sufficiently small norm $\|x\|$, and consequently for $u = P(t)x$, system (6) will be astmptotically stable independently of terms $g_i(t, x, u)$ in accordance with Liapunov's theorem [1]. Therefore control $u = P(t)x$ stabilizes the system. The theorem is proved.

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